

Self-similar groups and the zig-zag and replacement products of graphs

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September 1, 2014

Abstract

Every finitely generated self-similar group naturally produces an infinite sequence of finite d -regular graphs Γ_n . We construct self-similar groups, whose graphs Γ_n can be represented as an iterated zig-zag product and graph powering: $\Gamma_{n+1} = \Gamma_n^k \circledast \Gamma$ ($k \geq 1$). Also we construct self-similar groups, whose graphs Γ_n can be represented as an iterated replacement product and graph powering: $\Gamma_{n+1} = \Gamma_n^k \circledcirc \Gamma$ ($k \geq 1$). This gives simple explicit examples of self-similar groups, whose graphs Γ_n form an expanding family, and examples of automaton groups, whose graphs Γ_n have linear diameters $\text{diam}(\Gamma_n) = O(n)$ and bounded girth.

2010 Mathematics Subject Classification: 05C25, 20F65, 20E08

Keywords: self-similar group, zig-zag product, replacement product, expanding graph, automaton group

1 Introduction

A sequence of finite d -regular graphs $(\Gamma_n)_{n \geq 1}$ is an expanding family if there exists $\varepsilon > 0$ such that $\lambda(\Gamma_n) < 1 - \varepsilon$ for all $n \in \mathbb{N}$, where $\lambda(\Gamma)$ is the second largest (in absolute value) eigenvalue of the normalized adjacency matrix of Γ . Expanding graphs have many interesting applications in different areas of mathematics and computer science (see [11] and the references therein). That is why many constructions of expanding families were proposed for the last decades, most of which have algebraic nature.

In [16], Reingold, Vadhan, and Wigderson discovered a simple combinatorial construction of expanding graphs. Their construction is based on the new operation on regular graphs — the zig-zag product \circledast . The estimates on the second eigenvalue of the zig-zag product of graphs proved in [16] lead to the construction of expanders as an iterated zig-zag product and graph squaring: the sequence $\Gamma_{n+1} = \Gamma_n^2 \circledast \Gamma$, $\Gamma_1 = \Gamma^2$ is an expanding family if $\lambda(\Gamma)$ is small enough. Later, the zig-zag product showed its effectiveness in constructing graphs with other exceptional properties, various codes, in computational complexity theory, etc.

The zig-zag product is directly related to the simpler replacement product \textcircled{r} , which replace every vertex of one graph by a copy of another graph. This product was widely used in various contexts. For example, the replacement product of the graph of the d -dimensional cube and the cycle on d vertices is the so-called cube-connected cycle, which is used in the network architecture for parallel computations. Gromov [10] considered the graphs of d -dimensional cubes for different dimensions and estimated the second eigenvalue of their iterated replacement product (iterated cubical graphs). Previte [15] studied the convergence of iterated replacement product $\Gamma_{n+1} = \Gamma_n \textcircled{r} \Gamma$, normalized to have diameter one, in the Gromov-Hausdorff metric and their limit spaces. The estimate on the second eigenvalue of the replacement product of graphs proved in [16] leads to the expanding family $\Gamma_{n+1} = \Gamma_n^4 \textcircled{r} \Gamma$ when $\lambda(\Gamma)$ and $\lambda(\Gamma_1)$ are small enough [14].

In this paper we establish a connection between the zig-zag and replacement products of graphs and self-similar groups. The theory of self-similar groups [13] was developed from several examples of groups (mainly the Grigorchuk group) that enjoy many extreme properties (intermediate growth, finite width, non-uniformly exponential growth, periodic groups, amenable but not elementary amenable groups, just-infinite groups, etc.) Self-similar groups are specific groups of transformations on the space of all finite words over an alphabet that preserve the length of words. Every self-similar group can be easily defined by a finite system of wreath recursions, while properties of the group remain mysterious.

By fixing a generating set of a self-similar group, we get a sequence of d -regular graphs Γ_n associated to the action of generators on words of length n . A natural question arises whether we can produce an expanding family in this way. However, the graphs Γ_n were studied mostly for the opposite case of contracting self-similar groups. In this case, the graphs Γ_n converge in certain sense to a compact fractal space, which lead to the notion of a limit space of a contracting self-similar group and further developed into the beautiful theory of iterated monodromy groups [13]. The diameter of graphs Γ_n for contracting groups has exponential growth in terms of n (polynomial in the number of vertices), what makes them opposite to expanding graphs and the zig-zag product.

An important class of self-similar groups is the class of automaton groups. These groups are given by finite-state transducers (Mealy automata) with the same input and output alphabets. Every state of such an automaton A produces a transformation of words over an alphabet. If all these transformations are invertible, they generate a self-similar group under composition of functions called the automaton group G_A generated by A . For example, the Grigorchuk group is generated by a 5-state automaton over a 2-letter alphabet.

The graphs Γ_n for an automaton group G_A can be expressed through the standard operation of composition of automata, namely $\Gamma_n = \hat{A} \circ \dots \circ \hat{A}$ (n times), where \hat{A} is the dual automaton. However, expanding properties of automata composition are unknown. The complete spectrum of graphs Γ_n was computed only for a few automaton groups [3, 8, 9], and the general case remains widely open. Nevertheless, Glasner and Mozes [6] realized certain groups with property (T) as automaton groups, what implies that the associated graphs Γ_n form an expanding family. The corresponding automata are large and were not described explicitly. At the same time, there are two specific 3-state automata over a 2-letter alphabet, the Aleshin and Bellaterra automata, whose graphs Γ_n form asymptotic

expanders [7, Section 10], and the question is raised [7, Problem 10.1] whether actually these graphs are expanders. This problem remains open. Even the asymptotic of diameters of Γ_n for these two automata is unknown; the best known upper bound is $O(n^2)$ [12].

In this paper, given $k \geq 1$ and a graph Γ with certain restrictions, we construct self-similar groups, whose graphs Γ_n can be represented as iterated zig-zag or replacement products and graph powering: $\Gamma_{n+1} = \Gamma_n^k \circledast \Gamma$ for all $n \geq 1$ or $\Gamma_{n+1} = \Gamma_n^k \circledcirc \Gamma$ for all $n \geq 1$. This gives explicit examples of self-similar groups whose graphs Γ_n form a family of expanders. The established connection between self-similar groups and the zig-zag product is not surprising — the zig-zag product of graphs is closely related to the semidirect product of groups [1], while self-similar groups to the wreath product of groups. We also note that our construction modeling iterated zig-zag product is a self-similar analog of the construction from [17]. In the case $k = 1$, the constructed groups are automaton groups. This gives simple explicit examples of automaton groups whose graphs Γ_n have linear diameters $O(n)$ (logarithmic in the number of vertices) and bounded girth. Interestingly, some of the automaton groups modeling iterated replacement product belong to the class of GGS groups [2, 4]. In particular, these groups are not finitely presented and have intermediate growth.

2 The zig-zag and replacement products of graphs

All graphs in this paper are regular, undirected, and may have loops and multiple edges.

Let \mathcal{G} be a D -regular graph on N vertices and let Γ be a d -regular graph on D vertices. We label the edges near every vertex of \mathcal{G} by the vertices of Γ in one-to-one fashion; for $v \in V(\mathcal{G})$ and $x \in V(\Gamma)$, let $v[x]$ be the x -neighbor of v . If an edge is labeled by x near v and by y near u , i.e., $v[x] = u$ and $u[y] = v$, we write $v \xrightarrow{x \quad y} u$. The zig-zag and replacement products depend on the chosen labeling.

The zig-zag product. The *zig-zag product* $\mathcal{G} \circledast \Gamma$ is a d^2 -regular graph on ND vertices $V(\Gamma) \times V(\mathcal{G})$. The edges of $\mathcal{G} \circledast \Gamma$ are formed by “zig-zag” paths of length three:

1. for every edge $x - x'$ in Γ (the zig-step),
2. the edge $v \xrightarrow{x' \quad y'} u$ in \mathcal{G} ,
3. and every edge $y' - y$ in Γ (the zag-step),

there is an edge between (x, v) and (y, u) in $\mathcal{G} \circledast \Gamma$. (Classically, the vertices of the zig-zag product are written as pairs (v, x) . We switched the order to show a similarity with action graphs of self-similar groups. As usual, by switching from right to left, we get a connection between two object studied in different contexts.)

The next basic properties easily follow from the definition. The zig-zag product of any two graphs has girth ≤ 4 and diameter $\text{diam}(\mathcal{G} \circledast \Gamma) \leq \text{diam}(\mathcal{G}) + 2\text{diam}(\Gamma)$. The zig-zag product of connected graphs is not always connected; one easy sufficient condition is the following: If any two vertices of Γ can be connected by a path of even length, then the graph $\mathcal{G} \circledast \Gamma$ is connected for any connected graph \mathcal{G} .

Many applications of the zig-zag product are based on the following spectral property proved in [16]:

$$\lambda(\mathcal{G} \circledast \Gamma) \leq \lambda(\mathcal{G}) + \lambda(\Gamma) + \lambda(\Gamma)^2, \quad (1)$$

where $\lambda(\Gamma)$ is the second largest (in absolute value) eigenvalue of the normalized adjacency matrix of Γ .

The replacement product. The *replacement product* $\mathcal{G} \circledcirc \Gamma$ is a $(d+1)$ -regular graph on ND vertices $V(\Gamma) \times V(\mathcal{G})$ with the following edges:

1. for every edge $x - y$ in Γ and $v \in V(\mathcal{G})$ there is an edge $(x, v) - (y, v)$ in $\mathcal{G} \circledcirc \Gamma$;
2. for every edge $v \overset{x}{\text{---}} \overset{y}{\text{---}} u$ in \mathcal{G} there is an edge $(x, v) - (y, u)$ in $\mathcal{G} \circledcirc \Gamma$.

In other words, we replace each vertex v of \mathcal{G} with a copy of Γ (keeping all the edges of Γ in all the copies), and adjoin edges adjacent to v in \mathcal{G} to the corresponding vertices of Γ using the chosen one-to-one correspondence between these edges and vertices of Γ .

The next properties easily follow from the definition. The replacement product of connected graphs is connected, the diameter satisfies $\text{diam}(\mathcal{G} \circledcirc \Gamma) \leq \text{diam}(\mathcal{G}) \cdot \text{diam}(\Gamma)$, and the girth of $\mathcal{G} \circledcirc \Gamma$ is not greater than the girth of Γ .

In [16], the expansion property of the replacement product is estimated as

$$\lambda(\mathcal{G} \circledcirc \Gamma) \leq (p + (1 - p)(\lambda(\mathcal{G}) + \lambda(\Gamma) + \lambda(\Gamma)^2))^{1/3}, \quad (2)$$

where $p = d^2/(d+1)^3$.

Iterative construction of expanders. Let us describe the construction of expanding families using the zig-zag product and graph powering presented in [16]. Take a d -regular graph Γ on d^4 vertices such that $\lambda(\Gamma) \leq 1/5$ (such graphs exist by probabilistic arguments). Define the sequence of graphs $(\Gamma_n)_{n \geq 1}$ as follows:

$$\Gamma_1 = \Gamma^2, \quad \Gamma_{n+1} = \Gamma_n^2 \circledast \Gamma, \quad n \geq 1. \quad (3)$$

(The k -th power Γ^k of a graph Γ is the graph on the vertices of Γ , where each edge corresponds to a path of length k in Γ . Note that $\lambda(\Gamma^k) = \lambda(\Gamma)^k$.) Then the estimate (1) implies that the graphs Γ_n are d^2 -regular graphs with $\lambda(\Gamma_n) \leq 2/5$.

Analogous construction works with the replacement product as well [14]. Take a $(d+1)$ -regular graph Γ_1 and a d -regular graph Γ on $(d+1)^4$ vertices such that $\lambda(\Gamma_1) \leq 1/5$, $\lambda(\Gamma) \leq 1/5$. Define the sequence of graphs $(\Gamma_n)_{n \geq 1}$ as follows:

$$\Gamma_{n+1} = \Gamma_n^4 \circledcirc \Gamma, \quad n \geq 1. \quad (4)$$

The estimate (2) implies that the graphs Γ_n are $(d+1)$ -regular graphs with $\lambda(\Gamma_n) \leq 1/10$.

3 Self-similar groups and their action graphs

Every finitely generated self-similar group can be given by a finite system (wreath recursion)

$$\begin{cases} s_1 &= \pi_1(w_{11}, w_{12}, \dots, w_{1d}) \\ s_2 &= \pi_2(w_{21}, w_{22}, \dots, w_{2d}) \\ \vdots &\quad \quad \quad \vdots \\ s_k &= \pi_k(w_{k1}, w_{k2}, \dots, w_{kd}) \end{cases}, \quad (5)$$

where π_i is a permutation on $X = \{1, 2, \dots, d\}$ and w_{ij} is a word over $S \cup S^{-1}$, $S = \{s_1, s_2, \dots, s_k\}$. The system defines the action of S on the set X^* of all finite words over X (we use left actions). Each s_i acts on X by the permutation π_i , and the action on words over X is defined by the recursive rule

$$s_i(xv) = \pi_i(x)w_{ix}(v), \quad x \in X, v \in X^*,$$

where w_{ix} acts by composition. These transformations are invertible, and the group generated by them under composition is called the *self-similar group* $G = \langle S \rangle$ associated to the system (5).

When all words w_{ix} in (5) are letters, i.e., $w_{ix} = s_{ix} \in S$, the system (5) can be represented by a finite-state automaton-transducer A over the alphabet X with states S . The automaton A is represented by a finite directed graph with vertices S and arrows $s_i \rightarrow s_{ix}$ labeled by $x|\pi_i(x)$ for all $x \in X$ and $s_i \in S$. The action of s_i on X^* can be described using the automaton A as follows. Given a word $v = x_1x_2 \dots x_n \in X^*$, there exists a unique directed path in the automaton A starting at the state s_i and labeled by $x_1|y_1, x_2|y_2, \dots, x_n|y_n$ for some $y_i \in X$. Then the word $y_1y_2 \dots y_n$ is the image of $x_1x_2 \dots x_n$ under s_i . In this case, the group $G = \langle S \rangle$ is called the *automaton group* given by the automaton A .

Self-similar groups preserve the length of words under the action on X^* , and we can restrict the action to X^n , words of length n , for each $n \in \mathbb{N}$. By choosing a finite symmetric generating set S of a self-similar group G , we get a sequence of $|S|$ -regular graphs $(\Gamma_n)_{n \geq 1}$ of the action on X^n , $n \geq 1$. The vertex set of Γ_n is X^n , and for every $s \in S$ and $v \in X^n$ there is an edge between the vertices v and $s(v)$. The graph Γ_n is a Schreier coset graph of G if the group acts transitively on X^n .

If the generating set S is given by the system (5), the graphs $\Gamma_n = \Gamma_n(S)$ can be constructed iteratively, very similar to (3). Every $s_i \in S$ produces an edge $xv - \pi_i(x)w_{ix}(v)$ in Γ_n , which can be interpreted as a zig-step $x - \pi_i(x)$ in the graph Γ_1 , and a walk $v - w_{ix}(v)$ in the graph Γ_{n-1} . In contrast to the zig-zag product, we are missing the zag-step (and the walk w_{ix} is not agreed with $\pi_i(x)$), but we will see in the next section that one can make the edge $x - \pi_i(x)$ to be already the combination of the zig and zag steps.

4 Modeling iterated zig-zag and replacement products by automaton groups

In this section we construct automaton groups whose action graphs satisfy $\Gamma_{n+1} = \Gamma_n \mathbin{\textcircled{\small Z}} \Gamma$ for all $n \geq 1$ or $\Gamma_{n+1} = \Gamma_n \mathbin{\textcircled{\small R}} \Gamma$ for all $n \geq 1$, where Γ is a fixed graph.

Modeling iterated zig-zag product. Let $X = \{1, 2, \dots, d\}$ and P be a symmetric set of permutations on X such that $d = |P|^2$. We introduce formal symbols $s_{(\pi, \tau)}$ for $\pi, \tau \in P$ and define wreath recursion (5) for the set $S_P = \{s_{(\pi, \tau)} : \pi, \tau \in P\}$, $|S_P| = d$ as follows. Choose an order on S_P : let $S_P = \{s_1, \dots, s_d\}$. Let γ be the permutation on X given by the rule: if $s_x = s_{(\pi, \tau)}$ then $s_{\gamma(x)} = s_{(\tau^{-1}, \pi^{-1})}$. Notice that $\gamma = \gamma^{-1}$. Define wreath recursion by

$$s_{(\pi, \tau)} = \tau \gamma(s_1, s_2, \dots, s_d) \pi = \tau \gamma \pi(s_{\pi(1)}, s_{\pi(2)}, \dots, s_{\pi(d)}), \quad \pi, \tau \in P,$$

(here π and τ will play a role of the zig and zag steps respectively). Let G_P be the self-similar group defined by this recursion. It is important to note that S_P defines a symmetric generating set of G_P , where $s_{(\pi, \tau)}^{-1} = s_{(\tau^{-1}, \pi^{-1})}$ (in other notations, $s_x^{-1} = s_{\gamma(x)}$). This follows inductively from the recursions

$$\begin{aligned} s_{(\pi, \tau)}^{-1}(xv) &= \pi^{-1} \gamma^{-1} \tau^{-1}(x) s_{\gamma^{-1} \tau^{-1}(x)}^{-1}(v), \\ s_{(\tau^{-1}, \pi^{-1})}(xv) &= \pi^{-1} \gamma \tau^{-1}(x) s_{\tau^{-1}(x)}(v), \end{aligned}$$

$x \in X, v \in X^*$ (use $\gamma = \gamma^{-1}$).

Theorem 1. *The action graphs Γ_n of the group $G_P = \langle S_P \rangle$ satisfy $\Gamma_{n+1} = \Gamma_n \mathbin{\textcircled{\small Z}} \Gamma$, $n \geq 1$, where Γ is the graph of the action of P on X .*

Proof. The graph Γ is a $|P|$ -regular graph on d vertices, while Γ_n are d -regular graphs. In order to define the zig-zag product $\Gamma_n \mathbin{\textcircled{\small Z}} \Gamma$, we should label the edges of Γ_n by the vertices of Γ . For $x \in X$, define the x -neighbor of a vertex $v \in X^n$ as $v[x] := s_x(v)$. In this way we get the labeling of edges $v \xrightarrow{x \gamma(x)} s_x(v)$. Now we can consider the zig-zag product $\Gamma_n \mathbin{\textcircled{\small Z}} \Gamma$. The vertex set of $\Gamma_n \mathbin{\textcircled{\small Z}} \Gamma$ can be naturally identified with the vertex set X^{n+1} of Γ_{n+1} via $(x, v) \leftrightarrow xv$. For every zig-zag path

$$x \xrightarrow{\pi} x' = \pi(x) \text{ in } \Gamma, \quad v \xrightarrow{x' \gamma'} v[x'] \text{ in } \Gamma_n, \quad y' \xrightarrow{\tau} y = \tau(y') \text{ in } \Gamma$$

there is an edge $xv - yv[x']$ in $\Gamma_n \mathbin{\textcircled{\small Z}} \Gamma$. Here $y' = \gamma(x') = \gamma(\pi(x))$ and $v[x'] = s_{x'}(v) = s_{\pi(x)}(v)$. Therefore this edge is precisely the edge of Γ_{n+1} given by $s_{(\pi, \tau)}$:

$$xv - \tau(\gamma(\pi(x)))s_{\pi(x)}(v).$$

□

The next statement immediately follows from the properties of the zig-zag product.

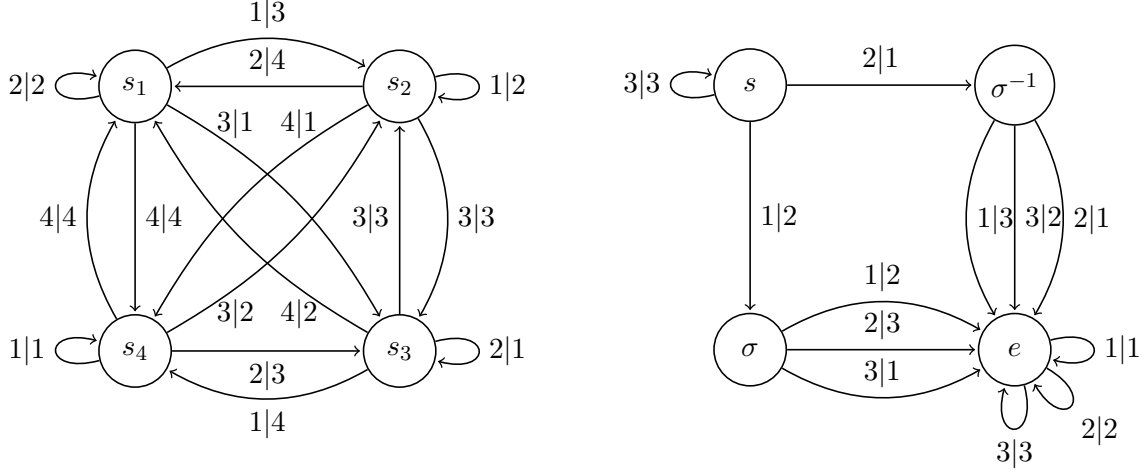


Figure 1: The generating automata for two examples of groups G_P and G_Q

Corollary 1.1. *The action graphs Γ_n of the group $G_P = \langle S_P \rangle$ have bounded girth and linear diameters $\text{diam}(\Gamma_n) = O(n)$. If Γ_1 is connected ($P\gamma P$ acts transitively on X) and there is a path of even length between any two vertices of Γ , then all graphs Γ_n are connected (the group G_P acts transitively on X^n).*

The wreath recursion for the group G_P defines a finite automaton over X with d states. Therefore every G_P is an automaton group.

Example 1. Let $d = 4$, $X = \{1, 2, 3, 4\}$ and $P = \{(1\ 2), (1\ 4)(2\ 3)\}$. Then $\gamma = (2\ 3)$ and the group G_P is generated by s_1, s_2, s_3, s_4 given by the wreath recursion:

$$\begin{aligned} s_1 &= (1\ 2)(2\ 3)(s_1, s_2, s_3, s_4)(1\ 2) = (1\ 3)(s_2, s_1, s_3, s_4) \\ s_2 &= (1\ 4)(2\ 3)(2\ 3)(s_1, s_2, s_3, s_4)(1\ 2) = (1\ 2\ 4)(s_2, s_1, s_3, s_4) \\ s_3 &= (1\ 2)(2\ 3)(s_1, s_2, s_3, s_4)(1\ 4)(2\ 3) = (1\ 4\ 2)(s_4, s_3, s_2, s_1) \\ s_4 &= (1\ 4)(2\ 3)(2\ 3)(s_1, s_2, s_3, s_4)(1\ 4)(2\ 3) = (2\ 3)(s_4, s_3, s_2, s_1) \end{aligned}$$

The generating automaton is shown on the left-hand side of Figure 1.

The construction can be modified for the case $d > |P|^2$. We add $d - |P|^2$ empty words e to the wreath recursion:

$$s_{(\pi, \tau)} = \tau \gamma(s_1, s_2, \dots, s_{|P|^2}, e, \dots, e) \pi,$$

where e acts trivially on X^* . Then the action graphs satisfy $\Gamma_{n+1} = \Gamma_n^\circ \otimes \Gamma$, where Γ_n° is obtained from Γ_n by adding $d - |P|^2$ loops to every vertex.

Modeling iterated replacement product. Let $Q = \{\pi_1, \dots, \pi_d\}$ be a symmetric set of permutations on $X = \{1, 2, \dots, d+1\}$. Let γ be the involution on X given by the rule:

$\pi_{\gamma(x)} = \pi_x^{-1}$ and $\gamma(d+1) = d+1$. We define wreath recursion for the set $S_Q = \{s_1, \dots, s_{d+1}\}$ by

$$\begin{aligned} s_i &= \pi_i(e, e, \dots, e), \quad i = 1, 2, \dots, d; \\ s_{d+1} &= \gamma(s_1, s_2, \dots, s_{d+1}). \end{aligned}$$

Let G_Q be the self-similar group defined by this recursion. Notice that the generating set S_Q is symmetric, because $s_i^{-1} = s_{\gamma(i)}$ for $i = 1, 2, \dots, d$ and $s_{d+1}^2 = e$.

Every s_i for $i = 1, \dots, d$ changes only the first letter in any word over X . In this case one usually identifies s_i and π_i ; then we can write $G_Q = \langle \pi_1, \dots, \pi_d, s \rangle$, where $s = s_{d+1}$ is given by the recursion $s = \gamma(\pi_1, \dots, \pi_d, s)$.

Theorem 2. *The action graphs Γ_n of the group $G_Q = \langle S_Q \rangle$ satisfy $\Gamma_{n+1} = \Gamma_n \mathbin{\textcircled{r}} \Gamma$, $n \geq 1$, where Γ is the graph of the action of Q on X . In particular, if Γ is connected, then all Γ_n are connected.*

Proof. The graph Γ is a d -regular graph on $d+1$ vertices, while Γ_n are $(d+1)$ -regular graphs. In order to define the replacement product $\Gamma_n \mathbin{\textcircled{r}} \Gamma$, we label the edges of Γ_n by the vertices of Γ as follows. For $x \in X$, define the x -neighbor of a vertex $v \in X^n$ as $v[x] := s_x(v)$. In this way we get the labeling of edges $v \xrightarrow{x \gamma(x)} s_x(v)$. Now we can consider the replacement product $\Gamma_n \mathbin{\textcircled{r}} \Gamma$. The vertex set of $\Gamma_n \mathbin{\textcircled{r}} \Gamma$ can be naturally identified with the vertex set X^{n+1} of Γ_{n+1} via $(x, v) \leftrightarrow xv$. For every edge $x \xrightarrow{\pi_i} y = \pi_i(x)$ in Γ , the edges $xv - yv$ in $\Gamma_n \mathbin{\textcircled{r}} \Gamma$ coincide with edges in Γ_{n+1} given by s_i . For every edge $v \xrightarrow{x \gamma(x)} s_x(v)$ in Γ_n , the edge $xv - \gamma(x)s_x(v)$ in $\Gamma_n \mathbin{\textcircled{r}} \Gamma$ coincide with the edge in Γ_{n+1} given by s_{d+1} . \square

The wreath recursion for the group G_Q defines a finite automaton over X with $d+2$ states. All these automata belong to the important class of bounded automata. In particular, the groups G_Q belong to the class of contracting self-similar groups (see [13]). The action graphs Γ_n of groups generated by bounded automata were studied in [5]. In particular, the diameters of graphs Γ_n have exponential growth in terms of n , and there is an algorithmic method to find the exponent of growth as the Perron-Frobenius eigenvalue of certain non-negative integer matrix.

Some of the groups G_Q were studied before as interesting examples of automaton groups. To see this, assume that all permutations in Q are involutions (then γ is trivial), $d \geq 2$, and the graph Γ is connected. Then the group G_Q is a GGS group studied in [2, 4]. In particular, in this case G_Q is not finitely presented and has intermediate growth. Is it true that all groups G_Q have subexponential growth?

Example 2. Let $d = 2$ and $Q = \{\sigma, \sigma^{-1}\}$, $\sigma = (1\ 2\ 3)$. The group G_Q is generated by σ and $s = (1\ 2)(\sigma, \sigma^{-1}, s)$. The generating automaton is shown on the right-hand side of Figure 1.

5 Wreath recursions leading to expanding graphs

In this section we construct wreath recursions that model the iterations (3) and (4).

Fix $k \geq 1$. Let $X = \{1, 2, \dots, d\}$ and P be a symmetric set of permutations on X such that $d = |P|^{2k}$. We introduce formal symbols $s_{(\pi, \tau)}$ for $\pi, \tau \in P$ and define wreath recursion for the set $S_P = \{s_{(\pi, \tau)} : \pi, \tau \in P\}$ as follows. Take all words of length k over S_P , there are $|S_P|^k = d$ such words, and fix an order on them: w_1, w_2, \dots, w_d . Let γ be the involution on X such that if $w_x = s_{(\pi_1, \tau_1)} s_{(\pi_2, \tau_2)} \dots s_{(\pi_k, \tau_k)}$ then $w_{\gamma(x)} = s_{(\tau_k^{-1}, \pi_k^{-1})} \dots s_{(\tau_2^{-1}, \pi_2^{-1})} s_{(\tau_1^{-1}, \pi_1^{-1})}$. Define wreath recursion by

$$s_{(\pi, \tau)} = \tau \gamma(w_1, w_2, \dots, w_d) \pi = \tau \gamma \pi(w_{\pi(1)}, w_{\pi(2)}, \dots, w_{\pi(d)}), \quad \pi, \tau \in P.$$

Let $G_{P,k}$ be the self-similar group generated by this recursion. Notice that S_P defines a symmetric generating set of $G_{P,k}$, where $s_{(\pi, \tau)}^{-1} = s_{(\tau^{-1}, \pi^{-1})}$. When $k = 1$ we get the groups $G_P = G_{P,1}$ from the previous section.

Let us consider the associated action graphs Γ_n . Note that each word w_x represents a path of length k in Γ_n , which is an edge in the graph Γ_n^k . For $x \in X$, define the x -neighbor of a vertex $v \in X^n$ in Γ_n^k as $v[x] := w_x(v)$. Then each edge $s_{(\pi, \tau)}(xv) = \tau(\gamma(\pi(x)))w_{\pi(x)}(v)$ in Γ_{n+1} is precisely the zig-zag path in $\Gamma_n^k \circledcirc \Gamma$. We get the following statement.

Theorem 3. *The action graphs Γ_n of the group $G_{P,k} = \langle S_P \rangle$ satisfy $\Gamma_{n+1} = \Gamma_n^k \circledcirc \Gamma$, $n \geq 1$, where Γ is the graph of the action of P on X .*

If $\lambda(\Gamma)$ and $\lambda(\Gamma_1)$ are small enough (for example, less than $1/5$), then we get a sequence of expanders. Therefore this construction gives simple explicit examples of self-similar groups whose graphs Γ_n form an expanding family.

As above the construction can be modified for the case $d > |P|^k$ by adding empty words to the wreath recursion.

Similarly we model the iteration (4). Fix $k \geq 1$. Let $Q = \{\pi_1, \dots, \pi_d\}$ be a symmetric set of permutations on $X = \{1, 2, \dots, (d+1)^k\}$. Let s be a formal symbol and $S_Q = \{\pi_1, \dots, \pi_d, s\}$, where π_i is considered as transformation of X^* that changes the first letter of words. Take all words of length k over S_Q , there are $(d+1)^k$ such words, and fix an order on them: $w_1, w_2, \dots, w_{(d+1)^k}$. Let γ be the involution on X such that if $w_x = s_1 s_2 \dots s_k$ then $w_{\gamma(x)} = s_k^{-1} \dots s_2^{-1} s_1^{-1}$, where for the symbol $s_i = s$ we put $s^{-1} := s$. We define the self-similar group $G_{Q,k} = \langle \pi_1, \dots, \pi_d, s \rangle$, where s is given by the wreath recursion

$$s = \gamma(w_1, w_2, \dots, w_{(d+1)^k}).$$

The set S_Q defines a symmetric generating set of $G_{Q,k}$, because Q is symmetric and $s^2 = e$.

As above we get the following statement.

Theorem 4. *The action graphs Γ_n of the group $G_{Q,k} = \langle S_Q \rangle$ satisfy $\Gamma_{n+1} = \Gamma_n^k \circledcirc \Gamma$, $n \geq 1$, where Γ is the graph of the action of Q on X .*

This construction produces other examples of self-similar groups whose graphs Γ_n form an expanding family when $k \geq 4$ and $\lambda(\Gamma)$ and $\lambda(\Gamma_1)$ are small enough.

Questions. It is interesting what are algebraic and geometric properties of the groups $G_{P,k}$ and $G_{Q,k}$. Are these groups finitely presented? have property (T)? What are their profinite completions? What are the properties of their action on the boundary of the space X^* , i.e., infinite sequences $x_1x_2 \dots$ over X ?

References

- [1] N. Alon, A. Lubotzky, and A. Wigderson, Semi-direct product in groups and zig-zag product in graphs: connections and applications (extended abstract), in: “42-nd IEEE Symposium on Foundations of Computer Science, Las Vegas, NV, 2001”, 630–637. IEEE Computer Society, Los Alamitos, CA, 2001.
- [2] L. Bartholdi, Croissance de groupes agissant sur des arbres. Ph.D. thesis, Université de Genève (2000)
- [3] L. Bartholdi and R. I. Grigorchuk, On the spectrum of Hecke type operators related to some fractal groups, Tr. Mat. Inst. Steklova 231 (2000), no. Din. Sist., Avtom. i Beskon. Gruppy, 5–45.
- [4] L. Bartholdi, R. Grigorchuk, Z. Šuník, Branch groups. In *Handbook of algebra*, vol. 3, pages 989–1112. North-Holland, Amsterdam, 2003.
- [5] I. Bondarenko, Groups generated by bounded automata and their Schreier graphs. Ph.D. dissertation, Texas A&M University (2007)
- [6] Y. Glasner, S. Mozes, Automata and square complexes, Geometriae Dedicata, Volume 111, 43–64 (2005)
- [7] R. I. Grigorchuk, Some topics in the dynamics of group actions on rooted trees, Tr. Mat. Inst. Steklova, Volume 273, 72–191 (2011)
- [8] R. Grigorchuk, A. Żuk, Spectral properties of a torsion-free weakly branch group defined by a three state automaton, Computational and statistical group theory (Las Vegas, NV/Hoboken, NJ, 2001), Contemp. Math., vol. 298, Amer. Math. Soc., Providence, RI, 2002, pp. 57–82.
- [9] R. Grigorchuk, Z. Šuník, Schreier spectrum of the Hanoi towers group on three pegs, Proceedings of Symposia in Pure Mathematics, Volume 77, 183–198 (2008)
- [10] M. Gromov, Filling Riemannian manifolds, J. Differential Geom., Volume 18, 1–147 (1983)

- [11] S. Hoory, N. Linial, A. Wigderson, Expander graphs and their applications, Bull. Amer. Math. Soc., Volume 43, 439–561 (2006)
- [12] A. Malyshev, I. Pak, Lifts, derandomization, and diameters of Schreier graphs of Mealy automata, Preprint (2014)
- [13] V. Nekrashevych, Self-similar groups. Mathematical Surveys and Monographs, vol.117, American Mathematical Society, Providence (2005)
- [14] C.A. Kelley, D. Sridhara, J. Rosenthal, Zig-zag and replacement product graphs and LDPC codes, Advances in Mathematics of Communications, Volume 2, No. 4, 347–372 (2008)
- [15] J.P. Previte, Graph substitutions, Ergodic Theory and Dynamical Systems, Vol.18, 661–685 (1998)
- [16] O. Reingold, S. Vadhan, and A. Wigderson, Entropy waves, the zig-zag graph product, and new constant-degree expanders, Ann. of Math. (2), Volume 155, 91–120 (2002)
- [17] E. Rozenman, A. Shalev, and A. Wigderson, Iterative construction of Cayley expander graphs, Theory of Computing, Volume 2, 91–120 (2006)